Generalized Vertex Algebras

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Abstract

We give a short introduction to generalized vertex algebras, using the notion of polylocal fields. We construct a generalized vertex algebra associated to a vector space $\mathfrak{h}$ with a symmetric bilinear form. It contains as subalgebras all lattice vertex algebras of rank equal to $\dim\mathfrak{h}$ and all irreducible representations of these vertex algebras.

1 Introduction

A *vertex algebra* is essentially the same as a chiral algebra in two-dimensional conformal field theory [BPZ, G, DMS]. In mathematics, vertex algebras arose naturally in the representation theory of infinite-dimensional Lie algebras and in the construction of the "moonshine module" for the Monster simple finite group [B, FLM].

Some of the most important vertex (super)algebras are the vertex (super)algebras V_Q associated to *integral lattices* Q [FK, B, FLM, K, LL]. If the lattice Q is not necessarily integral, one gets on V_Q the structure of a *generalized vertex algebra* as introduced in [FFR, DL, M] (under the name "vertex operator para-algebra" in [FFR]). This notion includes as special cases the notion of a "vertex superalgebra" (called just a "vertex algebra" in [K]) and the notion of a "colored vertex algebra" from [X]. The "parafermion algebras" of Zamolodchikov and Fateev [ZF1, ZF2] are also closely related to generalized vertex algebras (see [DL, Chapter 14]). The theory of generalized vertex algebras (and further generalizations) is developed in detail in the monograph [DL], and important examples of generalized vertex algebras are constructed in [FFR, DL, M, GL]. The treatment of [FFR, DL, M] is centered around a "Jacobi identity," which generalizes the Jacobi identity from [FLM] (the latter is equivalent to the Borcherds identity from [K]).

In the present paper we give a short introduction to generalized vertex algebras, utilizing the approach of [BN] (for $D = 1$) which is based on the notion

of *polylocal fields*. The notion of *locality* plays an important role in the theory of (generalized) vertex algebras (see [G, DL, K, Li1, GL, LL]). It is natural both from a physical and from a mathematical point of view to extend it to polylocality of fields in several variables. Our definition of a generalized vertex algebra is slightly more general than the ones from [FFR, DL, M] in that we allow a more general grading condition, and we do not assume an action of the Virasoro algebra (cf. [GL]). In particular, in [DL, M] only *rational lattices* Q give rise to generalized vertex algebras V_Q. In our setting, Q is allowed to be the whole *vector space*, which leads to a new generalized vertex algebra $V_\mathfrak{h}$ associated to any vector space $\mathfrak{h}$ with a symmetric bilinear form. It is remarkable that $V_\mathfrak{h}$ contains as subalgebras the lattice vertex (super)algebras V_Q for all integral lattices $Q \subseteq \mathfrak{h}$, as well as all of their irreducible representations.

The paper is organized as follows. In Sect. 2 we recall the notion of a *quantum field* and its generalization that corresponds to taking non-integral powers of the formal variable. We also introduce fields in several variables and (generalized) *polylocality*, following [BN].

In the presence of a grading by an abelian group Q, we define *parafermion fields* in Sect. 3.1. Then in Sect. 3.2 we show that a translation covariant, local and complete system of parafermion fields can be extended uniquely to a *state-field correspondence*. This result implies Uniqueness and Existence Theorems that generalize those of [G, FKRW, K] (see also [Li1, GL, LL]). The definition of a *generalized vertex algebra* is given in Sect. 3.3 in terms of (generalized) locality. In Sect. 3.4 we introduce a natural action of the *cohomology group* $\mathrm{H}^2(Q, \mathbb{C}^\times)$ on the isomorphism classes of generalized vertex algebras. We also show how in a special case the notion of a generalized vertex algebra reduces to that of a Q*-graded* vertex superalgebra. In Sect. 3.5 we discuss the *operator product expansion* of local parafermion fields, and we prove *formal associativity* and *commutativity* relations generalizing those of [FB, LL]. In Sect. 3.6 we derive the (generalized) *Jacobi identity* (= *Borcherds identity*), thus showing that our definition is a generalization of the ones from [FFR, DL, M]. The exposition of Sections 2, 3.2, 3.5 and 3.6 follows closely [BN].

In Sect. 4 we introduce the notion of a *module* over a generalized vertex algebra, following [DL]. We show that the notion of a *twisted module* over a vertex (super)algebra [FFR, D2] (cf. [DK]) is a special case of the notion of a module over a generalized vertex algebra, as observed in [Li3].

In Sect. 5 we construct the generalized vertex algebra $V_\mathfrak{h}$ associated to a *vector space* $\mathfrak{h}$ with a symmetric bilinear form, and we discuss its subalgebras V_Q associated to *integral lattices* $Q \subseteq \mathfrak{h}$.

2 Quantum fields in several variables and polylocality

2.1 Spaces of formal series

We first fix some notation to be used throughout the paper. Let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. By z, w, etc., we will denote formal commuting variables. All vector spaces are over the field $\mathbb{C}$ of complex numbers. We will denote

by V a vector space, and by $V[z]$ (respectively, $V[\![z]\!]$) the space of polynomials (respectively, formal power series) in z with coefficients in V.

We will identify the subsets $\Gamma \subseteq \mathbb{C}/\mathbb{Z}$ with subsets $\Gamma \subseteq \mathbb{C}$ that are $\mathbb{Z}$-*invariant*, i.e., that satisfy $\Gamma + \mathbb{Z} \subseteq \Gamma$. Given such a subset Γ, we denote by $V[\![z]\!]z^\Gamma$ the space of all finite sums of the form $\sum_i \psi_i(z) z^{d_i}$, where $d_i \in \Gamma$ and $\psi_i(z) \in V[\![z]\!]$. Another way to write the elements of $V[\![z]\!]z^\Gamma$ is as infinite sums $\sum_n f_n z^n$, where $f_n \in V$ and n runs over the union of finitely many sets of the form $\{d_i + \mathbb{Z}_+\}$ with $d_i \in \Gamma$. In particular, $V[\![z]\!]z^{\mathbb{Z}}$ is exactly the space $V(\!(z)\!) \equiv V[\![z]\!][z^{-1}]$ of formal Laurent series.

We denote by $V[\![z, z^\Gamma]\!]$ the space of all formal infinite series $\sum_{n\in\Gamma} f_n z^n$ with $f_n \in V$. For $\Gamma = \mathbb{Z}$ this coincides with the space $V[\![z, z^{-1}]\!]$ of formal power series in z, z^{-1}. Note that $V[\![z]\!]z^\Gamma$ is a $\mathbb{C}(\!(z)\!)$-module, while $V[\![z, z^\Gamma]\!]$ is only a $\mathbb{C}[z, z^{-1}]$-module, $V[\![z]\!]z^\Gamma$ being its submodule. In addition, both $V[\![z]\!]z^\Gamma$ and $V[\![z, z^\Gamma]\!]$ are equipped with the usual action of the derivative ∂_z.

In the same way, we introduce spaces of formal series in several variables $V[\![z_1, \dots, z_s]\!]z_1^{\Gamma_1} \cdots z_s^{\Gamma_s}$ and $V[\![z_1, z_1^{\Gamma_1}, \dots, z_s, z_s^{\Gamma_s}]\!]$. The latter consists of all infinite sums of the form

$$\sum_{n_j \in \Gamma_j} f_{n_1,\dots,n_s} z_1^{n_1} \cdots z_s^{n_s} \,, \qquad f_{n_1,\dots,n_s} \in V \,,$$

while the former is its subspace consisting of such sums with each n_j running over the union of finitely many sets of the form $\{d_{i,j} + \mathbb{Z}_+\}$ with $d_{i,j} \in \Gamma_j$. Let us point out that $V[\![z_1, z_2]\!]z_1^{\Gamma_1} z_2^{\Gamma_2}$ consists of series in which the powers of z_1 and z_2 are uniformly bounded from below. In contrast, elements of the space $\big(V[\![z_1]\!]z_1^{\Gamma_1}\big)[\![z_2]\!]z_2^{\Gamma_2}$ have powers of z_2 that are bounded from below but the powers of z_1 are possibly unbounded.

For $N \in \mathbb{C}$, we define the formal expansions

$$\begin{aligned} \iota_{z_1,z_2}(z_1 - z_2)^N &:= e^{-z_2\partial_{z_1}} z_1^N \\ &= \sum_{j\in\mathbb{Z}_+} \binom{N}{j} z_1^{N-j}(-z_2)^j \in \big(\mathbb{C}[\![z_1]\!]z_1^\Gamma\big)[\![z_2]\!] \,, \end{aligned} \tag{1}$$

$$\begin{aligned} \iota_{z_2,z_1}(z_1 - z_2)^N &:= e^{\pi \mathrm{i} N} e^{-z_1\partial_{z_2}} z_2^N \\ &= e^{\pi \mathrm{i} N} \sum_{j\in\mathbb{Z}_+} \binom{N}{j} (-z_1)^j z_2^{N-j} \in \big(\mathbb{C}[\![z_2]\!]z_2^\Gamma\big)[\![z_1]\!] \,, \end{aligned} \tag{2}$$

where $\Gamma = N + \mathbb{Z}$. Note that the spaces in the right-hand sides of Eqs. (1) and (2) are modules over the localized ring $\mathbb{C}[\![z_1, z_2]\!]_{z_1,z_2}$. More generally, it makes sense to multiply Eq. (1) by an element of $V[\![z_1, z_2]\!]z_1^{\Gamma_1} z_2^{\Gamma_2}$ thus producing an element of $\big(V[\![z_1]\!]z_1^{N+\Gamma_1}\big)[\![z_2]\!]z_2^{\Gamma_2}$, and similarly for Eq. (2). In this way, we extend the above expansions by linearity to maps

$$\begin{aligned} \iota_{z_1,z_2} &\colon V[\![z_1, z_2]\!]z_1^{\Gamma_1} z_2^{\Gamma_2} z_{12}^N \;\to\; \big(V[\![z_1]\!]z_1^{N+\Gamma_1}\big)[\![z_2]\!]z_2^{\Gamma_2} \,, \\ \iota_{z_2,z_1} &\colon V[\![z_1, z_2]\!]z_1^{\Gamma_1} z_2^{\Gamma_2} z_{12}^N \;\to\; \big(V[\![z_2]\!]z_2^{N+\Gamma_2}\big)[\![z_1]\!]z_1^{\Gamma_1} \,, \end{aligned}$$

where $z_{12} := z_1 - z_2$.

Obviously, when $N \in \mathbb{Z}_+$, expansions (1) and (2) are equal to each other and coincide with the binomial expansion of $(z_1 - z_2)^N$. Similarly, we define expansions of $(z_1 + z_2)^N$ for $N \in \mathbb{C}$. One has the following analog of *Taylor's formula*:

$$\iota_{z_1,z_2}\,\iota_{z_{12},z_2}\,f(z_{12} + z_2, z_2) = \iota_{z_1,z_2}\,f(z_1, z_2)\,, \quad z_{12} := z_1 - z_2\,, \tag{3}$$

for every localized series $f(z_1, z_2) = g(z_1, z_2)\, z_{12}^N$ with $N \in \mathbb{C}$ and $g(z_1, z_2) \in \mathbb{C}[\![z_1, z_2]\!] z_1^{\mathbb{C}} z_2^{\mathbb{C}}$. Indeed, it is enough to prove Eq. (3) for $f(z_1, z_2) = z_1^M$ with $M \in \mathbb{C}$, in which case it follows from (1) (see e.g. Proposition 2.2 from [BN] for more details).

2.2 Polylocal quantum fields

We define a *quantum field* in m variables $z_1, \dots, z_m$ (or just an m*-field* for short) to be a linear map from V to the space $V[\![z_1, \dots, z_m]\!] z_1^{\mathbb{C}} \cdots z_m^{\mathbb{C}}$. Alternatively, an m-field $A(z_1, \dots, z_m)$ can be viewed as a formal series from $(\mathrm{End}\, V)[\![z_1, z_1^{\mathbb{C}}, \dots, z_m, z_m^{\mathbb{C}}]\!]$ with the property that for every $v \in V$ one has:

$$A(z_1, \dots, z_m)v = \sum_i z_1^{d_{i,1}} \cdots z_m^{d_{i,m}}\, \psi_i(z_1, \dots, z_m)\,,$$

where the sum is finite, $d_{i,j} \in \mathbb{C}$ and $\psi_i \in V[\![z_1, \dots, z_m]\!]$.

If A is an m-field, then for every partition

$$\{1, \dots, m\} = J_1 \sqcup \cdots \sqcup J_r \qquad \text{(disjoint union)},$$

the *restriction*

$$\widetilde{A}(u_1, \dots, u_r)v := \big(A(z_1, \dots, z_m)v\big)\Big|_{z_j := u_s \ \text{for}\ j \in J_s} \tag{4}$$

makes sense and defines an r-field.

We will assume that V is endowed with an endomorphism T (called *translation operator*) and with a vector $|0\rangle$ (called *vacuum vector*), such that $T|0\rangle = 0$. An m-field A is called *translation covariant* if

$$T\,A(z_1, \dots, z_m) - A(z_1, \dots, z_m)\,T = \sum_{k=1}^{m} \partial_{z_k} A(z_1, \dots, z_m)\,.$$

Let us point out that a product $A(z_1, \dots, z_m)B(z_{m+1}, \dots, z_{m+n})$ of two fields is *not* a field in general. Indeed, by the above definition, for every $v \in V$ we have

$$\begin{aligned} A(z_1, \dots, z_m)v &\in V[\![z_1, \dots, z_m]\!] z_1^{\mathbb{C}} \cdots z_m^{\mathbb{C}}\,, \\ B(z_{m+1}, \dots, z_{m+n})v &\in V[\![z_{m+1}, \dots, z_{m+n}]\!] z_{m+1}^{\mathbb{C}} \cdots z_{m+n}^{\mathbb{C}}\,, \end{aligned}$$

which implies that $A(z_1, \dots, z_m)B(z_{m+1}, \dots, z_{m+n})v$ belongs to the space

$$\big(V[\![z_1, \dots, z_m]\!] z_1^{\mathbb{C}} \cdots z_m^{\mathbb{C}}\big)[\![z_{m+1}, \dots, z_{m+n}]\!] z_{m+1}^{\mathbb{C}} \cdots z_{m+n}^{\mathbb{C}}\,.$$

In general, elements of the latter space have unbounded powers of $z_1, \dots, z_m$. As a consequence, the restriction of the above product for coinciding arguments is not well defined in general. We will show below that one can "regularize" this product to make a field if the following definition is satisfied (see [BN]).

Definition 2.2. An m-field A and an n-field B are called *mutually local* if there exist complex numbers $N_{i,j}$ $(i = 1, \dots, m,\ j = 1, \dots, n)$ and $\eta \in \mathbb{C}^\times$ such that

$$\Bigl(\prod_{i=1}^{m} \prod_{j=m+1}^{m+n} \iota_{z_i,z_j}(z_i - z_j)^{N_{i,j}}\Bigr) A(z_1, \dots, z_m) B(z_{m+1}, \dots, z_{m+n}) \tag{5}$$
$$= \eta \Bigl(\prod_{i=1}^{m} \prod_{j=m+1}^{m+n} \iota_{z_j,z_i}(z_i - z_j)^{N_{i,j}}\Bigr) B(z_{m+1}, \dots, z_{m+n}) A(z_1, \dots, z_m)\,.$$

A 1-field that is local with respect to itself is usually called a *local* field; a 2-field that is local with respect to itself is called a *bilocal* field. An m-field, for general m, which is local with respect to itself, is called a *polylocal* field.

The following important result is a straightforward variation of Theorem 4.1 from [BN].

Proposition 2.2. *Let $A(z_1, \dots, z_m)$ and $B(z_{m+1}, \dots, z_{m+n})$ be an m-field and an n-field, respectively, which are mutually local as above.*

(a) *Every restriction of A for coinciding arguments is also a field and is mutually local with respect to B (see Eq.* (4)).

(b) *If the field A is translation covariant, then its restrictions for coinciding arguments are also translation covariant fields.*

(c) *If A is translation covariant, then $A(z_1, \dots, z_m)|0\rangle \in V[\![z_1, \dots, z_m]\!]$.*

(d) *Every partial derivative $\partial_{z_k} A$ is a field and is mutually local with respect to B. If the field A is translation covariant, then $\partial_{z_k} A$ is also translation covariant.*

(e) *The left-hand side $F_{A,B}$ of Eq.* (5) *is an $(m+n)$-field. If the fields A and B are local with respect to a p-field C, then $F_{A,B}$ is also local with respect to C. If both fields A and B are translation covariant, then $F_{A,B}$ is also translation covariant.*

Remark 2.2. The above Proposition remains valid if one considers a more general notion of polylocality, where in Eq. (5) the product $\prod (z_i - z_j)^{N_{i,j}}$ is replaced with some function of the differences $z_i - z_j$. One can replace that product with an even more general function if the translation covariance condition is modified accordingly. See also [Li4], where related ideas (in the case $m = n = 1$) were applied in the investigation of "quantum vertex algebras."

2.3 Operator product expansion

As a corollary of Proposition 2.2, every m-field $A(z_1, \dots, z_m)$ can be expanded in 1-fields as follows (see [BN]). Consider for $v \in V$ the formal expansion

$$\begin{aligned} &\iota_{z,w_1} \cdots \iota_{z,w_{m-1}} A(z+w_1, \dots, z+w_{m-1}, z)v \\ &\quad := \exp(w_1\partial_{z_1} + \cdots + w_{m-1}\partial_{z_{m-1}})\, A(z_1, \dots, z_m)v\big|_{z_1=\cdots=z_m=z} \\ &\quad \in \big(V[\![z]\!]z^{\mathbb{C}}\big)[\![w_1, \dots, w_{m-1}]\!]\,. \end{aligned} \tag{6}$$

This is a formal power series in $w_1, \dots, w_{m-1}$ with coefficients of the form $\psi_i(z)v \in V[\![z]\!]z^{\mathbb{C}}$ for some uniquely defined fields $\psi_i(z)$ (i running over some index set). All $\psi_i(z)$ are fields because they are obtained from $A(z_1, \dots, z_m)$ by the operations of differentiation and restriction. If, in addition, A is translation covariant and is local with respect to some other fields B, C, etc., then all the fields $\psi_i(z)$ are also translation covariant and local with respect to B, C, etc.

Formal expansion (6) is called the *operator expansion* of $A(z_1, \dots, z_m)$. Applying this expansion to the field $F_{A,B}$ given by the left-hand side of Eq. (5), we get what is called the *operator product expansion* (OPE) of two mutually local fields A and B. The following simple observation will be useful in the sequel.

Remark 2.3. It follows from Proposition 2.2(c) that for $v = |0\rangle$ the right-hand side of Eq. (6) is just the Taylor series expansion of

$$A(z+w_1, \dots, z+w_{m-1}, z)|0\rangle \in V[\![z, w_1, \dots, w_{m-1}]\!]\,.$$

This implies that the linear span of all coefficients of $A(z_1, \dots, z_m)|0\rangle$ coincides with the linear span of all coefficients of all $\psi_i(z)|0\rangle$, where $\{\psi_i(z)\}$ is the collection of fields appearing in operator expansion (6).

3 Generalized vertex algebras

3.1 Parafermion fields

Now we will introduce a grading on the vector space V and on the space of fields, which will allow us to make the notion of locality more concrete.

Let us fix a (not necessarily finite or discrete) abelian group Q, and let us assume that our vector space V is *Q-graded*: $V = \bigoplus_{\alpha \in Q} V_\alpha$. In addition, assume we are given a symmetric bilinear map $\Delta\colon Q \times Q \to \mathbb{C}/\mathbb{Z}$. As before, we will identify cosets $\Gamma \in \mathbb{C}/\mathbb{Z}$ with subsets of $\mathbb{C}$ of the form $d + \mathbb{Z}$ for some $d \in \mathbb{C}$. Note that, although $\Delta(\alpha, \beta)$ is defined mod $\mathbb{Z}$, $e^{-2\pi \mathrm{i} \Delta(\alpha,\beta)}$ is a well-defined complex number.

Definition 3.1. (a) A *parafermion field* (or just a *field* for short) of *degree* $\alpha \in Q$ is a formal series $a(z) \in (\operatorname{End} V)[\![z, z^{\mathbb{C}}]\!]$ with the property that

$$a(z)b \in V_{\alpha+\beta}[\![z]\!]z^{-\Delta(\alpha,\beta)} \qquad \text{for} \quad b \in V_\beta\,. \tag{7}$$

We denote by $\mathcal{F}_\alpha(V)$ the vector space of all fields of degree α, and by $\mathcal{F}(V) := \bigoplus_{\alpha\in Q}\mathcal{F}_\alpha(V)$ the vector space of all *fields* on V.

(b) Two parafermion fields $a(z)$, $b(z)$ of degrees α and β, respectively, are called *mutually local* if (see Eqs. (1), (2))

$$\iota_{z,w}(z-w)^N\, a(z)b(w) = \eta(\alpha,\beta)\,\iota_{w,z}(z-w)^N\, b(w)a(z) \tag{8}$$

for some $N \in \Delta(\alpha,\beta)$ and $\eta(\alpha,\beta)\in\mathbb{C}^\times$.

It is easy to see that Eq. (8) forces $N\in\Delta(\alpha,\beta)$ and the following relation:

$$\eta(\alpha,\beta)\,\eta(\beta,\alpha) = e^{-2\pi\mathrm{i}\Delta(\alpha,\beta)}\,, \qquad \alpha,\beta\in Q\,. \tag{9}$$

Then the above definition of locality is symmetric with respect to switching $a(z)$ and $b(z)$, due to Eqs. (1), (2), (9). The notion of locality can be extended to not necessarily homogeneous fields (i.e., of fixed degree) by requiring that all their homogeneous components be mutually local.

Example 3.1. For $Q=\mathbb{Z}/2\mathbb{Z}=\{\bar0,\bar1\}$, we can think of $V=V_{\bar0}\oplus V_{\bar1}$ as a *superspace*. Then the choice $\Delta(\alpha,\beta)=\mathbb{Z}$, $\eta(\alpha,\beta)=(-1)^{\alpha\beta}$ corresponds to the usual locality for fields on V (see [G, DL, K, Li1]).

Remark 3.1. Our notation coincides with that in [FFR, Chapter 0], except that in [FFR] the abelian group Q is denoted by Γ and is assumed finite. Let us compare our notation to that in [DL]. What is denoted by (α,β) in [DL] corresponds to our $\Delta(\alpha,\beta)$; however, in [DL] all (α,β) belong to a finite subgroup of $\mathbb{C}/2\mathbb{Z}$, while $\Delta(\alpha,\beta)$ take values in $\mathbb{C}/\mathbb{Z}$. The function $c(\alpha,\beta)$ from [DL] coincides with $\eta(\alpha,\beta)\,e^{\pi\mathrm{i}\Delta(\alpha,\beta)}$ in our notation; then the condition $c(\alpha,\beta)=c(\beta,\alpha)^{-1}$ is equivalent to Eq. (9) above. We prefer to work with $\eta(\alpha,\beta)$ instead of $c(\alpha,\beta)$ because $e^{\pi\mathrm{i}\Delta(\alpha,\beta)}$ is not well defined when $\Delta(\alpha,\beta)$ is defined mod $\mathbb{Z}$.

We are going to write fields in the conventional form

$$a(z)=\sum_{n\in\mathbb{C}} a_{(n)}z^{-n-1}\,, \qquad a_{(n)}\in\operatorname{End}V\,. \tag{10}$$

Then Eq. (7) is equivalent to the following conditions:

$$a_{(n)}b\in V_{\alpha+\beta} \quad\text{for}\quad a(z)\in\mathcal{F}_\alpha(V)\,,\;\; b\in V_\beta$$

and

$$a_{(n)}b=0 \quad\text{if}\quad n\notin\Delta(\alpha,\beta) \quad\text{or}\quad \operatorname{Re}n\gg 0\,. \tag{11}$$

The coefficients $a_{(n)}$ in Eq. (10) are called *modes* of $a(z)$.

Assume, in addition, that we are given the *vacuum vector* $|0\rangle\in V_0$ and *translation operator* $T\in\operatorname{End}V$ preserving the Q-grading of V and such that $T|0\rangle=0$. As before, a field $a(z)$ is called *translation covariant* if

$$Ta(z)-a(z)\,T=\partial_z a(z)\,.$$

In this case, Proposition 2.2(c) implies that $a(z)|0\rangle\in V[\![z]\!]$. On the other hand, the bilinearity of Δ implies $\Delta(\alpha,0)=\mathbb{Z}$ for all $\alpha\in Q$. Thus, every translation covariant parafermion field $a(z)$ of degree α satisfies $a(z)|0\rangle\in V_\alpha[\![z]\!]$.

3.2 Completeness and state-field correspondence

As in the previous subsection, let V be a Q-graded vector space endowed with a translation operator T and a vacuum vector $|0\rangle$.

Definition 3.2. A system of fields $\{\phi_i(z)\}_{i\in\mathcal{I}}$ is called *local* iff $\phi_i(z)$ and $\phi_j(z)$ are mutually local for every $i, j \in \mathcal{I}$. The system $\{\phi_i(z)\}$ is called *translation covariant* iff every $\phi_i(z)$ is translation covariant. Finally, the system $\{\phi_i(z)\}$ is called *complete* iff the coefficients of all formal series $\phi_{i_1}(z_1)\cdots\phi_{i_n}(z_n)|0\rangle$ ($n \in \mathbb{N}$) together with $|0\rangle$ span the whole vector space V.

The next result shows that, given a translation covariant, local and complete system of fields on V, one can extend it uniquely to a *state-field correspondence*.

Theorem 3.2. *Let $\{\phi_i(z)\}_{i\in\mathcal{I}}$ be a translation covariant, local and complete system of fields on V. Then for every $a \in V$ there exists a unique field, denoted as $Y(a,z)$, which is translation covariant, local with respect to all $\phi_i(z)$, and such that $Y(a,z)|0\rangle|_{z=0} = a$.*

The *proof* follows closely that of Theorem 4.2 from [BN]. Let us consider the vector space F of all translation covariant fields that are local with respect to $\phi_i(z)$ for all $i \in \mathcal{I}$. By Proposition 2.2(c), there is a well-defined linear map

$$\Phi\colon F \to V\,, \qquad \chi(z) \mapsto \chi(z)|0\rangle\big|_{z=0}\,.$$

Our goal is to show that this map is an isomorphism of vector spaces. Without loss of generality, we will assume that the system of fields $\{\phi_i(z)\}$ is *homogeneous*, i.e., that every field $\phi_i(z)$ has a certain degree α_i.

Consider for every fixed $m \in \mathbb{N}$ and $i_1, \dots, i_m \in \mathcal{I}$ the m-field

$$A(z_1,\dots,z_m) := \Big(\prod_{1\le k<l\le m} \iota_{z_k,z_l}(z_k - z_l)^{N_{kl}}\Big)\,\phi_{i_1}(z_1)\cdots\phi_{i_m}(z_m)\,,$$

where N_{kl} are the numbers fulfilling locality condition (8) for $\phi_{i_k}(z)$ and $\phi_{i_l}(z)$. Note that A is a translation covariant m-field and is local with respect to all $\phi_i(z)$, due to Proposition 2.2(e). Then all fields $\psi_i(z)$ appearing in the operator expansion of A are contained in F (see Eq. (6)). By Remark 2.3, all coefficients of $A(z_1,\dots,z_m)|0\rangle$ belong to the image of the above map Φ.

On the other hand, the product $\phi_{i_1}(z_1)\cdots\phi_{i_m}(z_m)|0\rangle$ belongs to the space $V[\![z_1]\!]z_1^{\mathbb{C}}\cdots[\![z_m]\!]z_m^{\mathbb{C}}$, which is a module over the algebra $\mathbb{C}[\![z_1]\!]z_1^{\mathbb{C}}\cdots[\![z_m]\!]z_m^{\mathbb{C}}$ (see Sect. 2.1 and 2.2). For $k < l$ each $\iota_{z_k,z_l}(z_k - z_l)^{N_{kl}}$ is invertible in the latter algebra, and we have

$$\phi_{i_1}(z_1)\cdots\phi_{i_m}(z_m)|0\rangle = \Big(\prod_{1\le k<l\le m} \iota_{z_k,z_l}(z_k - z_l)^{-N_{kl}}\Big)\,A(z_1,\dots,z_m)|0\rangle\,.$$

This implies that every coefficient of $\phi_{i_1}(z_1)\cdots\phi_{i_m}(z_m)|0\rangle$ can be expressed as a linear combination of coefficients of $A(z_1,\dots,z_m)|0\rangle$, and hence belongs to the image of Φ. Then completeness of the system $\{\phi_i(z)\}$ implies that the map Φ is surjective.

To prove injectivity, first notice that translation covariance implies $\chi(z)|0\rangle = e^{zT}(\chi(w)|0\rangle|_{w=0})$. In particular, every element $\chi(z)$ of the kernel of Φ satisfies $\chi(z)|0\rangle = 0$. Without loss of generality we can assume that $\chi(z)$ is a field of degree α for some $\alpha \in Q$. Then, by Proposition 2.2(e), $\chi(z)$ is local with respect to the field $A(z_1, \dots, z_m)$ (see Eq. (5)):

$$\Bigl(\prod_{l=1}^{m} \iota_{z,z_l}(z - z_l)^{N_l}\Bigr)\, \chi(z) A(z_1, \dots, z_m)$$
$$= \eta \,\Bigl(\prod_{l=1}^{m} \iota_{z_l,z}(z - z_l)^{N_l}\Bigr) A(z_1, \dots, z_m)\chi(z)$$

for some complex numbers N_l and η. If we apply both sides of this equation to the vacuum $|0\rangle$, the right-hand side becomes zero. Then the same argument as above will give that $\chi(z)$ vanishes on all coefficients of $A(z_1, \dots, z_m)|0\rangle$. Since these coefficients span V, we obtain that $\chi(z) = 0$. This completes the proof of the theorem. $\square$

From the proof of the above theorem one can deduce the following generalization of the Uniqueness Theorem from [G, K].

Corollary 3.2. *Let $\{\phi_i(z)\}$ be a local and complete system of fields on V, and let B be an n-field that is local with respect to all $\phi_i(z)$. Then if $B(z_1, \dots, z_n)$ annihilates the vacuum vector, it is identically zero on V.*

Remark 3.2. (cf. [DK]). It follows from the above Theorem that the collection of fields $\{Y(a,z)\}_{a\in V}$ coincides with the set of all parafermion fields that are translation covariant and local with respect to all $\phi_i(z)$.

3.3 Definition of generalized vertex algebra

Motivated by Theorem 3.2, we introduce the notion of a generalized vertex algebra in terms of locality in the spirit of [K] (cf. [G, DL, Li1, GL]). We will show in Sect. 3.6 below that our definition is a generalization of the original ones from [FFR, DL, M]. Fix an abelian group Q and a symmetric bilinear map $\Delta \colon Q \times Q \to \mathbb{C}/\mathbb{Z}$.

Definition 3.3. A *generalized vertex algebra* consists of the following **data**:

(*space of states*) a Q-graded vector space $V = \bigoplus_{\alpha\in Q} V_\alpha$;

(*vacuum vector*) a vector $|0\rangle \in V_0$;

(*translation operator*) a Q-grading preserving endomorphism $T \in \operatorname{End} V$;

(*state-field correspondence*) a Q-grading preserving linear map $Y \colon V \to \mathcal{F}(V)$, $a \mapsto Y(a,z) = \sum_{n\in\mathbb{C}} a_{(n)} z^{-n-1}$ from V to the space of parafermion fields on V,

subject to the following **axioms** ($a \in V_\alpha$, $b \in V_\beta$):

(*vacuum axiom*) $T|0\rangle = 0$, $Y(a,z)|0\rangle|_{z=0} = a$;

(*translation covariance*) $[T, Y(a,z)] = \partial_z Y(a,z)\,;$

(*locality*) $\iota_{z,w}(z-w)^N\, Y(a,z)Y(b,w)$
$$= \eta(\alpha,\beta)\,\iota_{w,z}(z-w)^N\, Y(b,w)Y(a,z)$$
for some $N \in \Delta(\alpha,\beta)$ and a bimultiplicative function $\eta\colon Q\times Q \to \mathbb{C}^\times$ such that $\eta(\alpha,\beta)\,\eta(\beta,\alpha) = e^{-2\pi \mathrm{i}\Delta(\alpha,\beta)}$.

A *homomorphism* of generalized vertex algebras is a linear map $f\colon V \to W$, preserving the Q-grading, mapping the vacuum vector to the vacuum vector, intertwining the actions of the translation operators, and preserving all n-th products: $f(a_{(n)}b) = f(a)_{(n)}f(b)$.

Remark 3.3. (cf. [DL]). One can define a more general notion of a homomorphism as a pair (f,φ), where $f\colon V \to W$ is a linear map and $\varphi\colon Q \to Q$ is a group homomorphism such that $\eta(\varphi\alpha,\varphi\beta) = \eta(\alpha,\beta)$. The map f should have the same properties as in the above Definition, except that instead of preserving the Q-grading it satisfies $f(V_\alpha) \subseteq W_{\varphi\alpha}$.

As a corollary of Theorem 3.2, we obtain a generalization of the Existence Theorem from [FKRW, K] (see also [Li1, GL, LL]).

Corollary 3.3. *Every translation covariant, local and complete system of parafermion fields $\{\phi_i(z)\}$ on a Q-graded vector space V generates on V a unique structure of a generalized vertex algebra.*

Examples 3.3. (a) For $Q = \{0\}$ the notion of a generalized vertex algebra coincides with that of a *vertex algebra*.

(b) For $Q = \mathbb{Z}/2\mathbb{Z}$ and $\eta(\alpha,\beta) = (-1)^{\alpha\beta}$, a generalized vertex algebra is the same as a *vertex superalgebra* (cf. Example 3.1).

(c) Let V be a generalized vertex algebra such that $\eta(\alpha,\beta) = (-1)^{p(\alpha)p(\beta)}$ for some homomorphism $p\colon Q \to \mathbb{Z}/2\mathbb{Z}$. Define a structure of a vector superspace on V by letting the parity of $a \in V_\alpha$ be $p(\alpha)$. Then V is a *vertex superalgebra* and it is *Q-graded*, i.e., $a_{(n)}b \in V_{\alpha+\beta}$ for all $a \in V_\alpha$, $b \in V_\beta$, $n \in \mathbb{Z}$.

We will give less obvious examples of generalized vertex algebras in Sect. 5 below; see [FFR, DL, GL] for additional examples. From now on, we will often use the notation $a(z) \equiv Y(a,z)$, and we will denote the modes of the field $Y(a,z)$ by $a_{(n)}$ as in Eq. (10). As already noticed before (see the proof of Theorem 3.2), translation covariance implies that
$$Y(a,z)|0\rangle = e^{zT}a\,, \qquad a \in V\,, \tag{12}$$
and in particular $Ta = a_{(-2)}|0\rangle$. This shows that the translation operator T is uniquely determined by the state-field correspondence Y. Another property, which follows from Theorem 3.2, is that $Y(Ta,z) = \partial_z Y(a,z)$. We also have the *skew-symmetry* relation [FFR, GL]:
$$Y(a,z)b = \eta(\alpha,\beta)\, e^{zT}\big(Y(b, e^{\pi \mathrm{i}}z)a\big)\,, \qquad a \in V_\alpha\,,\ b \in V_\beta\,. \tag{13}$$
This can be proved by applying both sides of Eq. (8) to $|0\rangle$, using (12) and translation covariance, and setting $w = 0$.

3.4 Action of the cohomology group $\mathrm{H}^2(Q,\mathbb{C}^\times)$

In this subsection we show that the second cohomology group $\mathrm{H}^2(Q,\mathbb{C}^\times)$ acts naturally on the isomorphism classes of generalized vertex algebras with fixed Q and Δ. To the best of our knowledge, this action is new; however, the idea of modifying the vertex operators by a 2-cocycle goes back to [FK].

Let Q be a fixed abelian group. It is well known that $\mathrm{H}^2(Q,\mathbb{C}^\times)$ parameterizes the *central extensions* of the group Q, i.e., exact sequences of group homomorphisms

$$1 \to \mathbb{C}^\times \to \widetilde{Q} \to Q \to 1 \tag{14}$$

such that the image of $\mathbb{C}^\times$ is central in $\widetilde{Q}$, up to equivalence. We start this subsection by reviewing the theory of such central extensions (see e.g. [Br, Chapter IV]). This material will be used here and in Sect. 5.2 below. As before, we will write the group operation in Q additively, while the one in $\widetilde{Q}$ will be written multiplicatively.

Given a section $e\colon Q \to \widetilde{Q}$ of extension (14), one can identify $\widetilde{Q}$ as a set with $\mathbb{C}^\times \times Q$, so that $e^\alpha := e(\alpha) \in \widetilde{Q}$ is identified with $(1,\alpha)$ for $\alpha \in Q$, and the embedding $\mathbb{C}^\times \to \widetilde{Q}$ is given by $c \mapsto (c,0)$. Then all elements of $\widetilde{Q}$ have the form $c\,e^\alpha \equiv (c,\alpha)$. The product in $\widetilde{Q}$ gives rise to a map $\varepsilon\colon Q\times Q \to \mathbb{C}^\times$ such that

$$e^\alpha e^\beta = \varepsilon(\alpha,\beta)\, e^{\alpha+\beta}\,, \qquad \alpha,\beta \in Q\,. \tag{15}$$

Associativity and unit properties of this product are equivalent to the condition that ε is a (normalized) 2-*cocycle*, i.e.,

$$\begin{gathered} \varepsilon(\alpha,\beta)\,\varepsilon(\alpha+\beta,\gamma) = \varepsilon(\beta,\gamma)\,\varepsilon(\alpha,\beta+\gamma)\,, \\ \varepsilon(\alpha,0) = \varepsilon(0,\alpha) = 1\,. \end{gathered} \tag{16}$$

If we choose a different section $e'\colon Q \to \widetilde{Q}$, then it is related to the section e by $e'(\alpha) = \rho(\alpha)\, e^\alpha$ for some function $\rho\colon Q \to \mathbb{C}^\times$. The 2-cocycle corresponding to the section e' is

$$\varepsilon'(\alpha,\beta) = \varepsilon(\alpha,\beta)\,\rho(\alpha)\,\rho(\beta)\,\rho(\alpha+\beta)^{-1}\,, \tag{17}$$

so it belongs to the same *cohomology class* as the 2-cocycle ε. The group (with respect to multiplication of functions) of all such cohomology classes is the *cohomology group* $\mathrm{H}^2(Q,\mathbb{C}^\times)$. This gives a one-to-one correspondence between equivalence classes of central extensions (14) and elements of $\mathrm{H}^2(Q,\mathbb{C}^\times)$.

For a given section $e\colon Q \to \widetilde{Q}$ as above, one finds that

$$e^\alpha e^\beta = \omega(\alpha,\beta)\, e^\beta e^\alpha\,, \qquad \alpha,\beta \in Q\,,$$

where

$$\omega(\alpha,\beta) = \varepsilon(\alpha,\beta)\,\varepsilon(\beta,\alpha)^{-1}\,.$$

Clearly, ω does not depend on the choice of section, so it depends only on the cohomology class of ε. Thus ω is an invariant of the central extension (14),

which we will call its *canonical invariant*. Introduce the group (with respect to multiplication) $\Omega(Q)$ of all bimultiplicative maps $\omega\colon Q \times Q \to \mathbb{C}^\times$ such that $\omega(\alpha,\alpha) = 1$ for all $\alpha \in Q$. Then the canonical invariant ω of any central extension (14) belongs to $\Omega(Q)$, and we get a group homomorphim from $\mathrm{H}^2(Q,\mathbb{C}^\times)$ to $\Omega(Q)$. The next result is well known.[1]

Lemma 3.4. *The above homomorphim from* $\mathrm{H}^2(Q,\mathbb{C}^\times)$ *to* $\Omega(Q)$ *is an isomorphism of groups. In particular, for every* $\omega \in \Omega(Q)$ *there exists a central extension* (14) *with a canonical invariant* ω.

Proof. It follows from, e.g., [Br, §V.6, Exercise 5] that the homomorphim $\mathrm{H}^2(Q,\mathbb{C}^\times) \to \Omega(Q)$ is surjective and the kernel is isomorphic to $\mathrm{Ext}_{\mathbb{Z}}(Q,\mathbb{C}^\times)$. But $\mathbb{C}^\times$ is isomorphic to $\mathbb{C}/\mathbb{Z}$ as a $\mathbb{Z}$-module. The latter is divisible, and hence is an injective $\mathbb{Z}$-module (see [Br, §III.4]). Therefore, $\mathrm{Ext}_{\mathbb{Z}}(Q,\mathbb{C}^\times) = \{0\}$. $\square$

Now let V be a generalized vertex algebra as in Definition 3.3. Given a 2-cocycle $\varepsilon\colon Q \times Q \to \mathbb{C}^\times$, we modify the state-field correspondence Y by defining

$$Y^\varepsilon(a,z)b := \varepsilon(\alpha,\beta)\, Y(a,z)b\,, \qquad a \in V_\alpha\,,\ b \in V_\beta\,. \tag{18}$$

Then Y^ε satisfies the vacuum and translation covariance axioms (with the same $|0\rangle$ and T) and the locality axiom with $\eta(\alpha,\beta)$ replaced by

$$\eta^\varepsilon(\alpha,\beta) := \eta(\alpha,\beta)\,\varepsilon(\beta,\gamma)\,\varepsilon(\alpha,\beta+\gamma)\,\varepsilon(\alpha,\gamma)^{-1}\,\varepsilon(\beta,\alpha+\gamma)^{-1}\,.$$

Using Eq. (16), this can be simplified to

$$\eta^\varepsilon(\alpha,\beta) := \eta(\alpha,\beta)\,\varepsilon(\alpha,\beta)\,\varepsilon(\beta,\alpha)^{-1} = \eta(\alpha,\beta)\,\omega(\alpha,\beta)\,, \tag{19}$$

where ω is the canonical invariant of the central extension defined by ε. Hence η^ε is bimultiplicative and satisfies Eq. (9) with the same Δ. Therefore, we obtain a new structure of a generalized vertex algebra on V with the same Q-grading, Δ, $|0\rangle$ and T but with modified Y and η.

Definition 3.4. The above-defined *modified* generalized vertex algebra will be denoted as V^ε. We say that two generalized vertex algebras V and W (for the same Q, Δ) are *equivalent* if W is isomorphic to V^ε for some 2-cocycle ε.

This is an equivalence relation because $V^{\varepsilon_1\varepsilon_2}$ is naturally isomorphic to $(V^{\varepsilon_1})^{\varepsilon_2}$. Note also that, for every homomorphism $f\colon V \to W$ of generalized vertex algebras, the same map f is a homomorphism from V^ε to W^ε (since f preserves the Q-grading). More generally, let $(f,\varphi)\colon V \to W$ be a homomorphism in the sense of Remark 3.3. Then for every 2-cocycle ε, $\varepsilon'(\alpha,\beta) := \varepsilon(\varphi\alpha,\varphi\beta)$ is also a 2-cocycle, and $(f,\varphi)\colon V^{\varepsilon'} \to W^\varepsilon$ is a homomorphism.

It is obvious from Eq. (19) that η^ε depends only on the cohomology class $[\varepsilon]$ of ε in $\mathrm{H}^2(Q,\mathbb{C}^\times)$. Given $\varepsilon' \in [\varepsilon]$ (see Eq. (17)), we define an endomorphism

[1] We thank Ofer Gabber for a discussion on this.

ρ of V by $\rho(a) := \rho(\alpha)a$ for $a \in V_\alpha$. Then (17) implies that $\rho\colon V^{\varepsilon'} \to V^\varepsilon$ is an isomorphism of generalized vertex algebras. We thus obtain an action of the cohomology group $\mathrm{H}^2(Q, \mathbb{C}^\times)$ on the isomorphism classes of generalized vertex algebras.

We finish this subsection with an important application of the above results.

Proposition 3.4. *Let V be a generalized vertex algebra for which the map $\Delta\colon Q \times Q \to \mathbb{C}/\mathbb{Z}$ is trivial, so that $\eta(\alpha, \alpha)^2 = 1$ for all $\alpha \in Q$. Then V is equivalent to a Q-graded vertex superalgebra, in which the parity of $a \in V_\alpha$ is $p(\alpha) \in \mathbb{Z}/2\mathbb{Z}$ where $(-1)^{p(\alpha)} = \eta(\alpha, \alpha)$.*

Proof. The function $\omega(\alpha, \beta) := (-1)^{p(\alpha)p(\beta)}\eta(\alpha, \beta)^{-1}$ is bimultiplicative and $\omega(\alpha, \alpha) = 1$ for all α. By Lemma 3.4, there exists a central extension (14) whose canonical invariant is ω. Let ε be a 2-cocycle defining this central extension. Then $\eta^\varepsilon(\alpha, \beta) = (-1)^{p(\alpha)p(\beta)}$, and hence V^ε is a Q-graded vertex superalgebra (see Example 3.3(c)). $\square$

3.5 Operator product expansion and associativity

In this subsection we investigate the operator product expansion of two parafermion fields in a generalized vertex algebra V (cf. Sect. 2.3).

Let $a \in V_\alpha$, $b \in V_\beta$ for some $\alpha, \beta \in Q$, and as before let $a(z) \equiv Y(a, z)$ for short. Denote by $F_{a,b}(z, w)$ the left-hand side of Eq. (8). By Proposition 2.2(e), $F_{a,b}(z, w)$ is a translation covariant bilocal field, which is local with respect to $c(z)$ for all $c \in V$. Then the *operator product expansion* (OPE) of $a(z)$ and $b(w)$ is defined by Eq. (6) for $A(z_1, z_2) = F_{a,b}(z_1, z_2)$, i.e., it is the expansion

$$\iota_{z,w}\, F_{a,b}(z + w, z) = e^{w\partial_{z_1}} F_{a,b}(z_1, z_2)\big|_{z_1=z_2=z} \,. \tag{20}$$

All coefficients in front of powers of w in this expansion are translation covariant fields and are local with respect to $c(z)$ for all $c \in V$ (see Proposition 2.2). Therefore, by Theorem 3.2, they must be themselves of the form $c(z)$ for some $c \in V$.

Lemma 3.5. *For every two elements a, b in a generalized vertex algebra, we have*:

$$\frac{1}{k!}\, \partial_{z_1}^k F_{a,b}(z_1, z_2)\big|_{z_1=z_2=z} = Y(a_{(N-1-k)}b, z)\,, \qquad k \in \mathbb{Z}_+\,, \tag{21}$$

where N is from Eq. (8) *and $F_{a,b}$ denotes the left-hand side of Eq.* (8).

Proof. By the above discussion, it is enough to establish Eq. (21) after both sides are applied to $|0\rangle$ and z is set to 0. But, by Proposition 2.2(c), $F_{a,b}(z_1, z_2)|0\rangle \in V[\![z_1, z_2]\!]$. Then $F_{a,b}(z_1, z_2)|0\rangle|_{z_2=0} = z_1^N a(z_1)b \in V[\![z_1]\!]$; hence applying to it $\frac{1}{k!}\,\partial_{z_1}^k$ and evaluating it at $z_1 = 0$ gives the coefficient in front of z_1^k in $z_1^N a(z_1)b$, which is exactly $a_{(N-1-k)}b$. $\square$

Let us point out that as a corollary of the above proof, we have $z^N a(z)b \in V[\![z]\!]$. Combining Eqs. (20) and (21), we obtain that the OPE of $a(z)$ and $b(w)$ is given by:

$$\iota_{z,w}\, F_{a,b}(z + w, z) = w^N\, Y(a(w)b, z)\,. \tag{22}$$

Denote by $N(a,b)$ the minimal $N \in \Delta(\alpha,\beta)$ (i.e., with minimal real part) fulfilling locality condition (8). Now we can prove the following *formal associativity* and *commutativity* relations generalizing those of [FB, LL] (see also [DL, BN]).

Theorem 3.5. *Let $a \in V_\alpha$, $b \in V_\beta$, $c \in V_\gamma$ $(\alpha,\beta,\gamma \in Q)$ be three elements in a generalized vertex algebra V. Then there exists a localized formal series*

$$\mathcal{Y}_{a,b,c}(z,w) = \psi_{a,b,c}(z,w)\, z^{-N(a,c)}\, w^{-N(b,c)}\, (z-w)^{-N(a,b)}\,,$$

where $\psi_{a,b,c}(z,w) \in V[\![z,w]\!]$, with the properties that

$$\begin{aligned} a(z)b(w)c &= \iota_{z,w}\, \mathcal{Y}_{a,b,c}(z,w)\,, \\ \eta(\alpha,\beta)\, b(w)a(z)c &= \iota_{w,z}\, \mathcal{Y}_{a,b,c}(z,w), \\ Y(a(w)b,z)c &= \iota_{z,w}\, \mathcal{Y}_{a,b,c}(z+w,z). \end{aligned}$$

Proof. We set $N = N(a,b)$ in Eq. (8), and define

$$\mathcal{Y}_{a,b,c}(z,w) := (z-w)^{-N(a,b)}\, F_{a,b}(z,w)c\,.$$

Then everything follows from Eqs. (8), (22) and the above observation that $z^{N(a,b)}\, a(z)b \in V[\![z]\!]$. $\square$

From this theorem we deduce the following *associativity* relations (see [DL, M, BK]).

Corollary 3.5. *For every three elements $a \in V_\alpha$, $b \in V_\beta$, $c \in V_\gamma$ $(\alpha,\beta,\gamma \in Q)$ in a generalized vertex algebra, and for each $L \in \Delta(\alpha,\gamma)$, $L \geq N(a,c)$, we have*:

$$\iota_{z,w}(z+w)^L\, \iota_{z,w} a(z+w)b(w)c = \iota_{w,z}(z+w)^L\, Y(a(z)b,w)c\,, \tag{23}$$

$$z^L\, a(z)b(w)c = \Big[\iota_{u,z-w}(u+z-w)^L\, \iota_{z,w} Y(a(z-w)b,u)c\Big]_{u=w}.$$

The expression under the substitution $u = w$ belongs to the space $u^{-N(b,c)} \times \iota_{z,w}(z-w)^{-N(a,b)}\, V[\![z,w,u]\!]$, and hence the substitution makes sense.

The *proof* is the same as that of Proposition 5.4 from [BN], utilizing Taylor's formula (3). $\square$

3.6 Jacobi identity and Borcherds identity

In this subsection we will show that every generalized vertex algebra V obeys the Jacobi identity of [FFR, DL, M], and hence our definition is a generalization of the ones from [FFR, DL, M].

First, recall that the *formal delta-function* is defined as the formal series (see Eqs. (1), (2)):

$$\delta(z,w) := (\iota_{z,w} - \iota_{w,z})(z-w)^{-1} = \sum_{j\in\mathbb{Z}} z^{-j-1} w^j\,. \tag{24}$$

More generally, for a subset $\Gamma \subseteq \mathbb{C}$, we define (see Sect. 2.1)

$$\delta_\Gamma(z,w) := \sum_{j\in\Gamma} z^{-j-1} w^j \in \mathbb{C}[\![z, z^\Gamma, w, w^\Gamma]\!]\,. \tag{25}$$

Note that if $\Gamma = d + \mathbb{Z}$ for some $d \in \mathbb{C}$, then $\delta_\Gamma(z,w) = (w/z)^d \delta(z,w)$, and this is independent of the choice of d mod $\mathbb{Z}$.

Theorem 3.6. *For every three elements $a \in V_\alpha$, $b \in V_\beta$, $c \in V_\gamma$ $(\alpha,\beta,\gamma \in Q)$ in a generalized vertex algebra, and for every $n \in \Delta(\alpha,\beta)$, we have the following identity*:

$$\begin{aligned} &Y(a,z)Y(b,w)c\ \iota_{z,w}(z-w)^n - \eta(\alpha,\beta)\, Y(b,w)Y(a,z)c\ \iota_{w,z}(z-w)^n \\ &\qquad = \sum_{j\in\mathbb{Z}_+} Y(a_{(n+j)}b,w)c\ \partial_w^j \delta_{\Delta(\alpha,\gamma)}(z,w)/j!\,. \end{aligned} \tag{26}$$

The sum in the right-hand side is finite due to Eq. (11)*.*

Proof. As in the proof of Theorem 5.5 from [BN], we deduce from Theorem 3.5 above that

$$\begin{aligned} &a(z)b(w)c\ \iota_{z,w}(z-w)^n \\ &\quad = z^{-L}\Big[\iota_{u,z-w}(u+z-w)^L\ \iota_{z,w}\,(z-w)^n\, Y(a(z-w)b,u)c\Big]_{u=w}\,, \end{aligned}$$

and

$$\begin{aligned} &\eta(\alpha,\beta)\, b(w)a(z)c\ \iota_{w,z}(z-w)^n \\ &\quad = z^{-L}\Big[\iota_{u,z-w}(u+z-w)^L\ \iota_{w,z}\,(z-w)^n\, Y(a(z-w)b,u)c\Big]_{u=w} \end{aligned}$$

for large enough $L \in \Delta(\alpha,\gamma)$. Recall that (see Eq. (1))

$$\iota_{u,z-w}(u+z-w)^L = \sum_{i\in\mathbb{Z}_+} \binom{L}{i} u^{L-i}\,(z-w)^i\,.$$

Take the difference of the above two expressions, and notice that

$$\begin{aligned} &(\iota_{z,w} - \iota_{w,z})\Big(\iota_{u,z-w}(u+z-w)^L\,(z-w)^n\, a(z-w)b\Big) \\ &\quad = \sum_{i\in\mathbb{Z}_+}\sum_{j\in\mathbb{Z}} \binom{L}{i} u^{L-i}\,(\iota_{z,w} - \iota_{w,z})(z-w)^{i-j-1}\, a_{(n+j)}b\,. \end{aligned}$$

Since $(\iota_{z,w} - \iota_{w,z})(z-w)^{i-j-1} = 0$ when $i-j-1 \geq 0$, the right-hand side can be rewritten as follows:

$$\sum_{j\in\mathbb{Z}_+} a_{(n+j)}b\ \sum_{i=0}^{j} \binom{L}{i} u^{L-i}\,(\iota_{z,w} - \iota_{w,z})(z-w)^{i-j-1}\,.$$

The sum over $j \in \mathbb{Z}_+$ is in fact finite, because $a_{(n+j)}b = 0$ for $j \gg 0$ by Eq. (11). Then both sums over i and j are finite.

Therefore, the substitution $u = w$ makes sense, and we obtain that the left-hand side of Eq. (26) is equal to

$$\sum_{j\in\mathbb{Z}_+} Y(a_{(n+j)}b, w)c \, \sum_{i=0}^{j} \binom{L}{i} z^{-L}\, w^{L-i}\, (\iota_{z,w} - \iota_{w,z})(z-w)^{i-j-1}\,.$$

Now observe that, by Eq. (24),

$$(\iota_{z,w} - \iota_{w,z})(z-w)^{-k-1} = \partial_w^{(k)}\delta(z,w)\,, \qquad k \in \mathbb{Z}_+\,,$$

where we use the divided-power notation $\partial_w^{(k)} := \partial_z^k/k!$. This formula and the fact that $\partial_w^{(i)} w^L = \binom{L}{i} w^{L-i}$, together with the Leibniz rule, imply that the sum over i in the above equation is exactly $\partial_w^{(j)}\big(z^{-L}\, w^L\, \delta(z,w)\big)$. Since, by definition, $z^{-L}w^L\delta(z,w) = \delta_{\Delta(\alpha,\gamma)}(z,w)$, this completes the proof. $\square$

We are going to rewrite Eq. (26) in terms of modes. By definition (see (10)), the modes of a field $a(z)$ are obtained from it by taking residues:

$$a_{(n)} = \mathrm{Res}_z\, z^n\, a(z)\,, \qquad n \in \mathbb{C}\,,$$

where Res_z gives the coefficient in front of z^{-1}. Now multiply both sides of Eq. (26) by $z^m w^k$, take $\mathrm{Res}_z\, \mathrm{Res}_w$, use expansions (1), (2), and note that, because of Eq. (11), only $m \in \Delta(\alpha,\gamma)$ and $k \in \Delta(\beta,\gamma)$ will give a nonzero result. We thus obtain the *Borcherds identity*:

$$\sum_{j\in\mathbb{Z}_+} (-1)^j \binom{n}{j} \Big(a_{(m+n-j)}(b_{(k+j)}c) - \eta(\alpha,\beta)\, e^{\pi i n}\, b_{(n+k-j)}(a_{(m+j)}c) \Big)$$
$$= \sum_{j\in\mathbb{Z}_+} \binom{m}{j} (a_{(n+j)}b)_{(m+k-j)}c\,, \tag{27}$$

for $a \in V_\alpha$, $b \in V_\beta$, $c \in V_\gamma$, $n \in \Delta(\alpha,\beta)$, $m \in \Delta(\alpha,\gamma)$, $k \in \Delta(\beta,\gamma)$.

It is remarkable that this identity has exactly the same form for (ordinary) vertex algebras, in which case $\eta(\alpha,\beta) = 1$ and $n, m, k \in \mathbb{Z}$ (see [K, (4.8.3)]). Borcherds used special cases of this identity in his original definition of the notion of a vertex algebra in [B].

Remark 3.6. Equation (27) above coincides with [FFR, (0.51)], which is equivalent to the “Jacobi identity” [FFR, (0.46)]. On the other hand, the collection of identities (26) for all $n \in \Delta(\alpha,\beta)$ is equivalent to the “Jacobi identity” [DL, (9.14)]. Indeed, the expression $u^{-1}\delta\big(\frac{z-w}{u}\big)$ used in [FLM, DL] coincides in our notation with $\iota_{z,w}\delta(z-w,u)$, while $u^{-1}\delta\big(\frac{w-z}{-u}\big)$ from [FLM, DL] is equal to our $\iota_{w,z}\delta(z-w,u)$. Thus, if we multiply Eq. (26) by u^{-n-1} and sum over $n \in \Delta(\alpha,\beta)$, we obtain [DL, (9.14)] with the corresponding changes in notation (cf. Remark 3.1).

We will now show that Borcherds identity (27), together with a "partial vacuum axiom" can be taken as an equivalent definition of a generalized vertex algebra.

Proposition 3.6. *Let Q be a fixed abelian group with a bimultiplicative map $\eta\colon Q \times Q \to \mathbb{C}^\times$, as in Sect.* 3.1. *Let V be a Q-graded vector space endowed with a vector $|0\rangle \in V_0$ and with products $a_{(n)}b \in V_{\alpha+\beta}$ for $n \in \mathbb{C}$, $a \in V_\alpha$, $b \in V_\beta$, satisfying Eqs.* (11), (27), *and*

$$|0\rangle_{(n)}a = \delta_{n,-1}\, a\,, \qquad a_{(-1)}|0\rangle = a\,.$$

Then V is a generalized vertex algebra with $Ta := a_{(-2)}|0\rangle$.

Proof. Since Eq. (27) is equivalent to (26), it implies locality of $a(z)$ and $b(z)$. The rest of the proof is as in Proposition 4.8(b) from [K]. $\square$

We refer to [FFR, DL, M, GL] for consequences of the Jacobi identity (= Borcherds identity), and we mention only one simple consequence here. From Eq. (26) and the sentence after Lemma 3.5, we deduce the following well-known fact.

Corollary 3.6. *For every two elements $a \in V_\alpha$, $b \in V_\beta$, the minimal number $N = N(a, b)$ (i.e., with minimal real part) fulfilling locality condition* (8) *is equal to the minimal $N \in \Delta(\alpha, \beta)$ such that $z^N a(z)b \in V[\![z]\!]$.*

4 Modules over generalized vertex algebras and twisted modules over vertex superalgebras

4.1 Modules over generalized vertex algebras

We start this subsection by recalling the notion of a module over a generalized vertex algebra, following [DL]. Fix an abelian group Q, a symmetric bilinear map $\Delta\colon Q \times Q \to \mathbb{C}/\mathbb{Z}$, and a generalized vertex algebra V, as in Sect. 3.3. Let P be a *Q-set*, i.e., a set with an action of Q on it, which we will write additively. Assume, in addition, that we are given an extension of Δ to a bilinear map $\Delta\colon Q \times P \to \mathbb{C}/\mathbb{Z}$. For instance, we may take $P = Q$ with the same Δ, but it will be convenient to work in the more general framework of Q-sets.

For a P-graded vector space $M = \bigoplus_{\gamma \in P} M_\gamma$, we define the notion of a (parafermion) field on M in the same way as in Definition 3.1(a). Namely, a *parafermion field* of *degree* $\alpha \in Q$ is a formal series $a(z) \in (\operatorname{End} M)[\![z, z^{\mathbb{C}}]\!]$ with the property that

$$a(z)c \in M_{\alpha+\gamma}[\![z]\!]z^{-\Delta(\alpha,\gamma)} \qquad \text{for} \quad c \in M_\gamma\,,\ \gamma \in P\,.$$

Then the vector space $\mathcal{F}(M)$ of all parafermion fields on M is again Q-graded.

Definition 4.1. A *module* over a generalized vertex algebra V (or a *V-module*) is a P-graded vector space $M = \bigoplus_{\gamma \in P} M_\gamma$ endowed with a Q-grading preserving linear map

$$Y\colon V \to \mathcal{F}(M)\,, \qquad a \mapsto Y(a, z) = \sum_{n \in \mathbb{C}} a_{(n)} z^{-n-1}\,,$$

from V to the space of parafermion fields on M, such that $Y(|0\rangle, z) = \mathrm{id}$ and Eq. (26) holds for all $a \in V_\alpha$, $b \in V_\beta$, $c \in M_\gamma$, $n \in \Delta(\alpha, \beta)$, ($\alpha, \beta \in Q$, $\gamma \in P$).

Remark 4.1. Equivalently, one can define the notion of a V-module in terms of Borcherds identity (27). One can also replace the Borcherds identity (or Jacobi identity) in the definition of a module by associativity relation (23). This was proved in [Li1] in the case of vertex algebras but the same proof works also for generalized vertex algebras, by making use of skew-symmetry relation (13) (cf. [Li2] and Remark 4.2 below).

Example 4.1. Let V be a generalized vertex algebra and let M be a V-module, as above. Assume that

$$\eta(\alpha, \beta) = (-1)^{p(\alpha)p(\beta)}\,, \quad \Delta(\alpha, \gamma) = \mathbb{Z}\,, \qquad \alpha, \beta \in Q,\ \gamma \in P\,,$$

for some homomorphism $p\colon Q \to \mathbb{Z}/2\mathbb{Z}$. Then V is a Q-graded vertex superalgebra (see Example 3.3(c)), and M is a P*-graded* V-module, i.e., $a_{(n)}c \in M_{\alpha+\gamma}$ for $a \in V_\alpha$, $c \in M_\gamma$, $n \in \mathbb{Z}$.

The following special situation provides important examples of modules.

Proposition 4.1. *Let P be an abelian group endowed with a symmetric bilinear map $\Delta\colon P \times P \to \mathbb{C}/\mathbb{Z}$, let V be a generalized vertex algebra for P, and let Q be a subgroup of P. For a coset $\Gamma \in P/Q$ (considered as a subset of P), introduce the subspace $V_\Gamma := \bigoplus_{\gamma\in\Gamma} V_\gamma$ of V. Then V_Q is a generalized vertex algebra for the group Q, and each V_Γ is a V_Q-module.*

The *proof* is immediate from the definitions. $\square$

From this Proposition and the results of Sect. 3.4, we deduce the following corollary, which can be used to construct modules over vertex superalgebras (see Sect. 5.2 below).

Corollary 4.1. *In the setting of the above Proposition, assume in addition that P/Q is isomorphic to a direct product of cyclic groups and $\Delta(\alpha, \gamma) = \mathbb{Z}$ for all $\alpha \in Q$, $\gamma \in P$. Then there exists a 2-cocycle $\varepsilon\colon P \times P \to \mathbb{C}^\times$ such that V_Q^ε is a Q-graded vertex superalgebra and each V_Γ^ε is a Γ-graded V_Q^ε-module.*

Proof. By Proposition 3.4 and its proof, there exists an element $\omega \in \Omega(Q)$ such that V_Q^ε is a vertex superalgebra, for every 2-cocycle $\varepsilon\colon Q \times Q \to \mathbb{C}^\times$ with canonical invariant ω. Let us choose representatives $\gamma_i \in P$ for the generators of the cyclic factors of P/Q, where i runs over some (possibly infinite) index set. Then we extend ω to a bimultiplicative map $P \times P \to \mathbb{C}^\times$ by letting $\omega(\gamma_i + \alpha, \gamma_j + \beta) = \omega(\alpha, \beta)$ for all i, j and all $\alpha, \beta \in Q$. Then $\omega \in \Omega(P)$, and hence it corresponds to a 2-cocycle $\varepsilon\colon P \times P \to \mathbb{C}^\times$ (see Lemma 3.4). $\square$

4.2 Twisted modules over vertex superalgebras

Let us recall the definition of a twisted module over a vertex (super)algebra [FFR,D2], in the formulation given in [DK]. Fix an additive subgroup $\Gamma \subseteq \mathbb{C}/\mathbb{Z}$,

and assume that V is a Γ-*graded* vertex superalgebra, i.e., $V = \bigoplus_{\alpha\in\Gamma} V_\alpha$ so that $a_{(n)}b \in V_{\alpha+\beta}$ for all $a \in V_\alpha$, $b \in V_\beta$, $n \in \mathbb{Z}$ (cf. Example 3.3(c)).

Definition 4.2. A Γ-*twisted module* over V is a vector space M endowed with a linear map

$$Y : V \to \mathcal{F}(M)\,, \qquad a \mapsto Y(a,z) = \sum_{n\in\mathbb{C}} a_{(n)} z^{-n-1}\,,$$

from V to the space of parafermion fields on M, such that $Y(|0\rangle, z) = \mathrm{id}$,

$$Y(a,z)c \in M[\![z]\!]z^{-\alpha} \quad \text{for} \quad a \in V_\alpha\,,\ c \in M\,,\ \alpha \in \Gamma\,, \tag{28}$$

and the following identity holds (cf. Eq. (25)):

$$\begin{aligned} Y(a,z)Y(b,w)c\ \iota_{z,w}(z-w)^n - (-1)^{p(a)p(b)}\, Y(b,w)Y(a,z)c\ \iota_{w,z}(z-w)^n \\ = \sum_{j\in\mathbb{Z}_+} Y(a_{(n+j)}b,w)c\ \partial_w^j \delta_\alpha(z,w)/j! \end{aligned} \tag{29}$$

for all $a \in V_\alpha$, $b \in V$, $c \in M$, $n \in \mathbb{Z}$, $\alpha \in \Gamma$, where a and b have parities $p(a)$ and $p(b)$, respectively.

Example 4.2. Let σ be a *diagonalizable* automorphism of V, and let Γ be a subgroup of $\mathbb{C}/\mathbb{Z}$ such that all eigenvalues of σ have the form $e^{2\pi \mathrm{i}\alpha}$ with $\alpha \in \Gamma$. For example, when $\sigma^N = 1$, we can take $\Gamma = \frac{1}{N}\mathbb{Z}/\mathbb{Z}$. Then V is Γ-graded by the eigenspaces of σ: $a \in V_\alpha$ iff $\sigma a = e^{2\pi\mathrm{i}\alpha}a$. For this Γ-grading, Eq. (28) is equivalent to the "monodromy" condition $Y(a,z) = Y(\sigma a, e^{2\pi \mathrm{i}}z)$. Then the above notion of a Γ-twisted module coincides with the notion of a σ-*twisted module* from [FFR, D2].

Remark 4.2. Written in terms of modes, Eq. (29) becomes again Borcherds identity (27). In the definition of a twisted module, the Borcherds identity (or Jacobi identity) can be replaced by associativity relation (23) (see [Li2]).

Comparing Eqs. (26) and (29), it is obvious that the notion of a twisted module is a special case of the notion of a module over a generalized vertex algebra. This can be stated more precisely as follows (cf. [Li3]).

Proposition 4.2. *Let V be a Γ-graded vertex superalgebra, where Γ is an additive subgroup of $\mathbb{C}/\mathbb{Z}$. Introduce the abelian group $Q = (\mathbb{Z}/2\mathbb{Z}) \times \Gamma$ and a Q-grading on V by*

$$a \in V_{(p,\alpha)} \quad \textit{iff} \quad p(a) = p\,,\ a \in V_\alpha\,,$$

where $p(a)$ denotes the parity of a. Let $P = \{\gamma_0\}$ be a one-element set with the trivial action of Q, and let

$$\eta(\hat\alpha,\hat\beta) = (-1)^{pq}\,, \quad \Delta(\hat\alpha,\hat\beta) = \mathbb{Z}\,, \quad \Delta(\hat\alpha,\gamma_0) = \alpha\,,$$

for $\hat\alpha = (p,\alpha)$, $\hat\beta = (q,\beta) \in Q$. Then a module over V as a generalized vertex algebra is the same as a Γ-twisted module over V as a vertex superalgebra.

5 Generalized vertex algebras associated to additive subgroups of vector spaces

5.1 Generalized vertex algebras associated to vector spaces

Let $\mathfrak{h}$ be a vector space (over $\mathbb{C}$) equipped with a symmetric bilinear form $(\cdot|\cdot)$, which is not necessarily non-degenerate. Inspired by the construction in [FK] and the construction of lattice vertex algebras [B, FLM, K, LL], we construct a generalized vertex algebra $V_\mathfrak{h}$ as follows.

Let $\hat{\mathfrak{h}} = \mathfrak{h}[t, t^{-1}] \oplus \mathbb{C}K$ be the *Heisenberg algebra*; this is a Lie algebra with the bracket

$$[\alpha t^m, \beta t^n] = m\delta_{m,-n}(\alpha|\beta)K\,, \quad [\alpha t^m, K] = 0\,.$$

The Heisenberg algebra has a representation with $K = 1$ on the *Fock space* $F := S(\mathfrak{h}[t^{-1}]t^{-1})$ (symmetric algebra), so that elements of $\mathfrak{h}[t^{-1}]t^{-1}$ act by multiplication and $\mathfrak{h}[t]1 = 0$. (This representation is irreducible iff the bilinear form $(\cdot|\cdot)$ is non-degenerate.) Let $\mathbb{C}[\mathfrak{h}]$ be the *group algebra* of $\mathfrak{h}$ considered as an additive group; as a vector space $\mathbb{C}[\mathfrak{h}]$ has a basis $\{e^\alpha\}_{\alpha\in\mathfrak{h}}$, and the multiplication is $e^\alpha e^\beta := e^{\alpha+\beta}$.

Our generalized vertex algebra $V_\mathfrak{h}$ will be the $\mathfrak{h}$-graded vector space

$$V_\mathfrak{h} := F \otimes_\mathbb{C} \mathbb{C}[\mathfrak{h}] = \bigoplus_{\alpha\in\mathfrak{h}} V_\alpha\,, \quad V_\alpha = F \otimes e^\alpha\,.$$

We take $Q = \mathfrak{h}$ (as an additive group), and $\Delta(\alpha, \beta) = -(\alpha|\beta) + \mathbb{Z}$. Next, we will define certain parafermion fields on $V_\mathfrak{h}$. For every $\alpha \in \mathfrak{h}$, consider the expression

$$E_\alpha(z) := \exp\Bigl(\sum_{n>0} \frac{1}{n}\alpha_{-n} z^n\Bigr) \exp\Bigl(\sum_{n<0} \frac{1}{n}\alpha_{-n} z^n\Bigr)\,,$$

where $\alpha_n \in \operatorname{End} F$ denotes the linear operator representing $\alpha t^n \in \hat{\mathfrak{h}}$. It is easy to see that $E_\alpha(z)$ is a well-defined field on F, i.e., it is a linear map from F to $F(\!(z)\!)$. We define the *vertex operators*

$$Y_\alpha(z) := E_\alpha(z) \otimes e^\alpha z^\alpha \colon V_\mathfrak{h} \to V_\mathfrak{h}[\![z]\!]z^\mathbb{C}\,,$$

where e^α acts on $\mathbb{C}[\mathfrak{h}]$ by multiplication and z^α by $z^\alpha e^\beta := z^{(\alpha|\beta)} e^\beta$. Then

$$Y_\alpha(z)v \in V_{\alpha+\beta}(\!(z)\!) z^{(\alpha|\beta)}\,, \qquad v \in V_\beta\,,$$

which means that $Y_\alpha(z)$ is a parafermion field on $V_\mathfrak{h}$ of degree α (see Definition 3.1(a)). We extend the action of the Heisenberg algebra from F to $V_\mathfrak{h}$ by letting $\alpha_n(v \otimes e^\beta) := (\alpha_n v) \otimes e^\beta + \delta_{n,0}(\alpha|\beta)v \otimes e^\beta$. Then we introduce the *currents*

$$\alpha(z) := \sum_{n\in\mathbb{Z}} \alpha_n\, z^{-n-1} \in (\operatorname{End} V_\mathfrak{h})[\![z, z^{-1}]\!]\,,$$

which are fields on $V_{\mathfrak{h}}$ of degree 0.

Proposition 5.1. *Introduce the vacuum vector* $|0\rangle := 1 \otimes e^0 \in V_{\mathfrak{h}}$ *and the translation operator* $T \in \operatorname{End} V_{\mathfrak{h}}$ *determined uniquely by*

$$T(1 \otimes e^\alpha) := \alpha t^{-1} \otimes e^\alpha \,, \quad [T, \alpha_n] = -n\, \alpha_{n-1} \,.$$

Let $Q = \mathfrak{h}$ (*as an additive group*), *and let*

$$\Delta(\alpha,\beta) = -(\alpha|\beta) + \mathbb{Z} \,, \quad \eta(\alpha,\beta) = e^{\pi \mathrm{i} (\alpha|\beta)} \,. \tag{30}$$

Then the collection of parafermion fields

$$Y(\alpha t^{-1} \otimes e^0, z) = \alpha(z) \,, \quad Y(1 \otimes e^\alpha, z) = Y_\alpha(z) \qquad (\alpha \in \mathfrak{h})$$

is translation covariant, local and complete, and hence it defines a unique structure of a generalized vertex algebra on $V_{\mathfrak{h}}$.

The *proof* is the same as for lattice vertex algebras, using Corollary 3.3 (see [K, Sect. 5.4]), and is in fact simpler because of the absence of the 2-cocycle ε. Let us recall here only the following crucial formulas, which imply locality:

$$\begin{aligned}
[\alpha(z), \beta(w)] &= (\alpha|\beta)\, \partial_w \delta(z,w) \,, \qquad\qquad \alpha,\beta \in \mathfrak{h} \,, \\
[\alpha(z), Y_\beta(w)] &= (\alpha|\beta)\, Y_\beta(w)\, \delta(z,w) \,, \\
Y_\alpha(z)\, Y_\beta(w) &= \iota_{z,w} (z-w)^{(\alpha|\beta)} E_{\alpha,\beta}(z,w) \otimes e^{\alpha+\beta} z^\alpha w^\beta \,,
\end{aligned}$$

where

$$E_{\alpha,\beta}(z,w) = \exp\Bigl(\sum_{n>0} \frac{1}{n} \bigl(\alpha_{-n} z^n + \beta_{-n} w^n \bigr) \Bigr) \exp\Bigl(\sum_{n<0} \frac{1}{n} \bigl(\alpha_{-n} z^n + \beta_{-n} w^n \bigr) \Bigr)$$

and $\delta(z,w)$ is the formal delta-function (24). $\square$

As an immediate corollary of Propositions 4.1 and 5.1, we obtain a generalized vertex algebra $V_Q = F \otimes_{\mathbb{C}} \mathbb{C}[Q]$, associated to any additive subgroup $Q \subseteq \mathfrak{h}$, and generated by the collection of parafermion fields $\{\alpha(z), Y_\beta(z)\}_{\alpha \in \mathfrak{h}, \beta \in Q}$. When the bilinear form $(\cdot|\cdot)$ is non-degenerate, the algebra V_Q is *simple* (i.e., it does not have nontrivial proper ideals), because then the Fock space F is an irreducible $\hat{\mathfrak{h}}$-module. In the case when Q is a *rational lattice* (i.e., a free abelian group of finite rank with a non-degenerate symmetric bilinear form over $\mathbb{Q}$), the generalized vertex algebra V_Q was constructed in [DL, M].

5.2 Vertex superalgebras associated to integral additive subgroups of vector spaces

In this subsection we will continue to use the notation of Sect. 5.1, and will assume in addition that $\mathfrak{h}$ is finite dimensional. We fix an additive subgroup $Q \subseteq \mathfrak{h}$, which is *integral*, i.e., such that $(\alpha|\beta) \in \mathbb{Z}$ for all $\alpha,\beta \in Q$. Let P be the set of all $\gamma \in \mathfrak{h}$ such that $(\alpha|\gamma) \in \mathbb{Z}$ for all $\alpha \in Q$. Then P is an additive

subgroup of $\mathfrak{h}$ containing Q, and the factor group P/Q is isomorphic to a direct product of a vector space and a finite abelian group.

By Propositions 4.1 and 5.1, we have a generalized vertex algebra $V_Q = F \otimes_{\mathbb{C}} \mathbb{C}[Q]$ and a family of V_Q-modules $V_{\gamma+Q} = F \otimes_{\mathbb{C}} e^{\gamma}\mathbb{C}[Q]$ for every coset $\gamma + Q \in P/Q$ (where $\gamma \in P \subseteq \mathfrak{h}$). The integrality assumption implies that $\Delta(\alpha, \gamma) \in \mathbb{Z}$ for all $\alpha \in Q$, $\gamma \in P$ (see Eq. (30)). Thus we can apply Corollary 4.1 to obtain a Q-graded vertex superalgebra V_Q^{ε} and a family of V_Q^{ε}-modules $V_{\gamma+Q}^{\varepsilon}$, for some 2-cocycle $\varepsilon\colon P \times P \to \mathbb{C}^{\times}$.

Let us spell out this construction in more detail, utilizing the results of Sect. 3.4. First of all, recall that as a vector space V_Q^{ε} coincides with V_Q. We define a structure of a vector superspace on it by letting the parity of $a \in V_\alpha$ be $p(\alpha) := (\alpha|\alpha) \bmod 2\mathbb{Z}$; then $\eta(\alpha, \alpha) = (-1)^{p(\alpha)}$. We choose a 2-cocycle $\varepsilon\colon Q \times Q \to \mathbb{C}^{\times}$ such that $\eta^{\varepsilon}(\alpha, \beta) = (-1)^{p(\alpha)p(\beta)}$. It corresponds to a central extension (14) with a canonical invariant (cf. Eqs. (19), (30))

$$\omega(\alpha, \beta) := (-1)^{(\alpha|\alpha)(\beta|\beta)+(\alpha|\beta)}, \qquad \alpha, \beta \in Q .$$

Since Q is a free abelian group, such ε can be constructed explicitly, for example as in [K, Remark 5.5a]. It follows from the results of Sect. 3.4 that, up to isomorphism, V_Q^{ε} does not depend on the choice of a 2-cocycle ε.

The vertex superalgebra V_Q^{ε} is generated by the following fields (cf. Eq. (18) and Proposition 5.1):

$$\begin{aligned} &Y^{\varepsilon}(\alpha t^{-1} \otimes e^0, z) = \alpha(z)\,, \qquad \alpha \in \mathfrak{h} \qquad\qquad (\textit{currents}),\\ &Y^{\varepsilon}(1 \otimes e^{\alpha}, z) = Y_\alpha(z)\, c_\alpha\,, \qquad \alpha \in Q \qquad (\textit{vertex operators}),\end{aligned}$$

where $c_\alpha|_{V_\beta} := \varepsilon(\alpha, \beta)\, \mathrm{id}_{V_\beta}$. Alternatively, we can define $Y^{\varepsilon}(1 \otimes e^{\alpha}, z)$ as $Y_\alpha(z)$ but replace everywhere the group algebra $\mathbb{C}[Q]$ with the *ε-twisted group algebra* $\mathbb{C}_\varepsilon[Q]$, which coincides with $\mathbb{C}[Q]$ as a vector space but has multiplication defined by Eq. (15).

Remark 5.2. If the bilinear form $(\cdot|\cdot)$ is non-degenerate, then Q is an *integral lattice* and P is a lattice of the same rank as Q, called its *dual lattice*. Then V_Q^{ε} coincides with the lattice vertex (super)algebra from [B, FLM, K, LL]. It is a simple vertex (super)algebra, and the V_Q^{ε}-modules V_Γ^{ε} $(\Gamma \in P/Q)$ are irreducible. These modules were constructed in [FLM] by a different method (see also [LL]). It was proved in [D1] that they form a complete list of irreducible V_Q^{ε}-modules up to isomorphism.

Acknowledgments

The first author wishes to thank Nikolay M. Nikolov for inspiring discussions and for collaboration on [BN]. Bakalov was supported in part by an FRPD grant from North Carolina State University. Kac was supported in part by NSF grant DMS-0501395.

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